ABSOLUTE TAUBERIAN CONDITIONS FOR ABSOLUTE HAUSDORFF AND QUASI-HAUSDORFF METHODS

BY

D. LEVIATAN

ABSTRACT

The purpose of this paper is to prove that for a large set of absolute Hausdorff and quasi-Hausdorff methods the condition

$$\sum_{k=1}^{\infty} \left| \lambda_n a_n - \lambda_{n-1} a_{n-1} \right| < \infty$$

is a Tauberian condition, i.e., its fulfillment together with the absolute summability of $\sum_{n=0}^{\infty} a_n$ to s implies that $\sum_{n=0}^{\infty} \left| a_n \right| < \infty$ and $\sum_{n=0}^{\infty} a_n = s$.

1. Introduction. A sequence $\{s_n\}$ $(s_n = a_0 + \cdots + a_n)$ is said to be absolutely summable by a method $T = \| \tau_{nk} \|$, or $|T|$-summable if the sequence

$$\sigma_n = \sum_{k=0}^{\infty} \tau_{nk} s_k$$

is absolutely convergent, that is $\sum_{n=0}^{\infty} \left| \sigma_{n+1} - \sigma_n \right| < \infty$. A condition on the a_n is called an absolute Tauberian condition for T if its fulfillment by the $a_n = s_n - s_{n-1}$ together with the $|T|$-summability of $\{s_n\}$ implies that $\{s_n\}$ is absolutely convergent to the same limit.

We shall write $a_n = \Omega(c_n)$, $c_n > 0$, if the sequence $\{a_n/c_n\}$ is absolutely convergent; and as a convenient reference we state here Lorentz [7] theorem 1. The theorem is conveniently stated for a series to sequence method T given by

$$\sigma_m = \sum_{k=0}^{\infty} b_{mk} a_k. \tag{1.1}$$

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LORENTZ'S THEOREM. *Let $c_n > 0$ be a bounded sequence and $m = m(n)$ a function increasing to infinity with n such that the sequences*

$$\beta_\nu(n) = \sum_{k=\nu}^{\infty} b_{mk}c_k - \sum_{k=\nu}^{n} c_k, \qquad n = 0, 1, 2, \cdots, \tag{1.2}$$

(the second sum being zero for $n < \nu$) have uniform bounded variations in the variable n for $\nu = 0, 1, 2, \cdots$, that is,

$$\sup_{\nu \geqq 0} \sum_{n=0}^{\infty} \left| \beta_\nu(n+1) - \beta_\nu(n) \right| < \infty. \tag{1.3}$$

Then $a_n = \Omega(c_n)$ is an absolute Tauberian condition for T.

In Lorentz's paper [7], theorem 1 is formulated with the restrictive condition $a_n/c_n \to 0$ by mistake, and Lorentz's intention (private communication) was to state it without this condition. The proof does not use this condition at all.

Let the sequence $\{\lambda_n\}$ $(n \geqq 0)$ satisfy

$$0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \to \infty, \quad \sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \infty.$$

The generalized Hausdorff transformation by means of the moments $\{\mu_n\}$ $(n \geqq 0)$, or in short $[H; \mu_n]$, of the sequence $\{s_n\}$ $(n \geqq 0)$ is defined (see [1]) by

$$\sigma_n = \sum_{k=0}^{n} \lambda_{nk} s_k \qquad n = 0, 1, 2, \cdots \tag{1.4}$$

where

$$\lambda_{nk} = (-1)^{n-k} \lambda_{k+1} \cdot \cdots \cdot \lambda_n \sum_{i=k}^{n} \mu_i / \omega'_{nk}(\lambda_i), \qquad 0 \leqq k < n = 1, 2, \cdots,$$

$$\lambda_{nn} = \mu_n, \qquad n = 0, 1, 2, \cdots$$

and $\omega_{nk}(x) = (x - \lambda_k) \cdot \cdots \cdot (x - \lambda_n)$, $0 \leqq k \leqq n = 0, 1, 2, \cdots$.

It is known ([1]) that $[H; \mu_n]$ is regular if and only if the moments possess the representation

$$\mu_n = \int_0^1 t^{\lambda_n} d\alpha(t) \qquad n = 0, 1, 2, \cdots, \tag{1.5}$$

where $\alpha(t)$ is a function satisfying

(1.6) $\alpha(t)$ is of bounded variation in $[0,1]$, $\alpha(0) = \alpha(0+) = 0$ and $\alpha(1) = 1$.

First we characterize the absolutely regular Hausdorff methods, i.e., those transforming every absolutely convergent sequence into an absolutely convergent sequence converging to the same limit.

THEOREM 1. *The method $[H;\mu_n]$ is absolutely regular if and only if it is regular.*

For the ordinary Hausdorff methods Theorem 1 is due partially to Knopp and Lorentz [4] and partially to Ramanujan [9].

An absolutely regular Hausdorff method is thus given by means of a function $\alpha(t)$ satisfying (1.6) and will be denoted by $H(\alpha)$.

We shall restrict ourselves to functions $\alpha(t)$ which satisfy the additional condition

$$\int_0^1 \frac{|\alpha(t)|}{t}\,dt < \infty \tag{1.7}$$

We propose the following result.

THEOREM 2. *Let the function $\alpha(t)$ satisfy* (1.6) *and* (1.7). *Then $\lambda_n a_n = \Omega(1)$ is an absolute Tauberian condition for $H(\alpha)$.*

For the sequence $\lambda_n = n$, $n \geqq 0$, and for the function $\alpha(t) = 1-(1-t)^\alpha$, $\alpha > 0$, the method $H(\alpha)$ reduces to Cesàro method (C,α). Theorem 2 in this case is due to Hyslop [2].

The generalized quasi-Hausdorff transformation by means of the moments $\{\mu_n\}$ (n $\geqq 0$), or in short $[QH;\mu_n]$, of the series $\sum_{n=0}^{\infty}{}' a_n$ is defined (see [5]) by

$$\sigma_n = \sum_{k=0}^{n} \tau_k, \quad \tau_k = \sum_{i=k}^{\infty} \lambda_{ik} a_i \tag{1.8}$$

It is known (see [5]) that $[QH;\mu_n]$ is regular if and only if the moments possess the representation (1.5) where $\alpha(t)$ is of bounded variation in $[0,1]$ and $\alpha(1)-\alpha(0) = 1$.

We characterize the absolutely regular quasi-Hausdorff methods by

THEOREM 3. *The method $[QH;\mu_n]$ is absolutely regular if and only if it is regular.*

Again we see that an absolutely regular quasi-Hausdorff method is given by means of some $\alpha(t)$ and so will be denoted by $QH(\alpha)$.

We propose here the analog of Theorem 2, namely,

THEOREM 4. *Let the function* $\alpha(t)$ *satisfy* (1.6) *and* (1.7). *Then* $\lambda_n a_n = \Omega(1)$ *is an absolute Tauberian condition for* $QH(\alpha)$.

2. Proofs.

PROOF OF THEOREM 1. It follows by (1.4) that

$$\sigma_n = \sum_{i=0}^{n} a_i \sum_{k=i}^{n} \lambda_{nk}.$$

Therefore by Knopp and Lorentz [4] Satz 2 necessary and sufficient conditions in order that $[H;\mu_n]$ should be absolutely regular are

$$\sup_{i\geqq 0} \sum_{n=i}^{\infty} \left| \sum_{k=i}^{n} [\lambda_{nk} - \lambda_{n-1,k}] \right| = M < \infty \tag{2.1}$$

(where $\lambda_{n-1,n} = 0$ $n = 0,1,2,\cdots$) and

$$\lim_{n\to\infty} \sum_{k=0}^{n} \lambda_{nk} = 1, \ \lim_{n\to\infty} \lambda_{nk} = 0 \qquad k = 0,1,2,\cdots. \tag{2.2}$$

It was shown by Hausdorff [1] (14) that for $0 \leqq k < n$

$$\lambda_{nk} - \lambda_{n-1,k} = \frac{\lambda_k}{\lambda_n}\lambda_{n_i} - \frac{\lambda_{k+1}}{\lambda_n}\lambda_{n,k+1}. \tag{2.3}$$

Consequently (2.1) becomes

$$\sup_{i\geqq 0} \sum_{n=i}^{\infty} \frac{\lambda_i}{\lambda_n} |\lambda_{n_i}| = M < \infty$$

which by [5] theorem 3.1, is equivalent to the sequence $\{\mu_n\}$ $(n \geqq 0)$ being represented by (1.5) where $\alpha(t)$ is of bounded variation in $[0,1]$. By [1] (7), (25) and Satz 1 it follows that (2.2) is now equivalent to $\alpha(0) = \alpha(0+) = 0$ and $\alpha(1) = 1$ and this completes our proof.

For convenience denote

$$p_{nk}(t) = (-1)^{n-k}\lambda_{k+1}\cdot\cdots\cdot\lambda_n \sum_{i=k}^{n} t^{\lambda_i}/\omega'_{nk}(\lambda_i) \qquad 0 \leqq k < n = 1,2,\cdots$$

$$p_{nn}(t) = t^{\lambda_n}, \qquad n = 0,1,2,\cdots$$

where $\omega_{nk}(x)$ is defined after (1.4).

Then if $\{\mu_n\}$ $(n \geqq 0)$ possesses the representation (1.5), then

$$\lambda_{nk} = \int_0^1 p_{nk}(t)d\alpha(t) \qquad 0 \leqq k \leqq n = 0, 1, 2, \cdots. \tag{2.4}$$

Proof of Theorem 2. It follows by (1.4), (2.4) and [5] p. 46 (11) that

$$\begin{aligned} \sigma_n &= \sum_{i=0}^{n} a_i \sum_{k=i}^{n} \int_0^1 p_{nk}(t)d\alpha(t) \\ &= a_0 + \sum_{i=1}^{n} a_i \sum_{k=i}^{n} \int_0^1 p_{nk}(t)d\alpha(t). \end{aligned} \tag{2.5}$$

Therefore if $\nu' = \max\{1, \nu\}$ and if $c_n = (1/\lambda_n)$, $n \geqq 1$, we have

$$\begin{aligned} \beta_\nu(n) &= \sum_{i=\nu'}^{n} \frac{1}{\lambda_i} \sum_{k=i}^{n} \int_0^1 p_{nk}(t)d\alpha(t) - \sum_{i=\nu'}^{n} \frac{1}{\lambda_i} \\ &= -\sum_{i=\nu'}^{n} \frac{1}{\lambda_i} \sum_{k=0}^{i-1} \int_0^1 p_{nk}(t)d\alpha(t). \end{aligned} \tag{2.6}$$

Now by [3] (3.9)

$$\frac{1}{\lambda_i} \sum_{k=0}^{i-1} p_{nk}(t) = \int_t^1 u^{-1} p_{ni}(u)du, \qquad 1 \leqq i \leqq n,$$

whence

$$\begin{aligned} \frac{1}{\lambda_i} \int_0^1 \sum_{k=0}^{i-1} p_{nk}(t)d\alpha(t) &= \int_0^1 \int_t^1 u^{-1} p_{ni}(u)dud\alpha(t) \\ &= \int_0^1 u^{-1} p_{ni}(u) \int_0^u d\alpha(t)du \\ &= \int_0^1 \frac{\alpha(u)}{u} p_{ni}(u)du. \end{aligned} \tag{2.7}$$

Thus by (2.6) and (2.7)

$$\beta_\nu(n) = -\sum_{i=\nu'}^{n} \int_0^1 \frac{\alpha(u)}{u} p_{ni}(u)du. \tag{2.8}$$

It follows by (2.3) that

$$\beta_\nu(n) - \beta_\nu(n+1) = \sum_{i=\nu'}^{n+1} \int_0^1 \frac{\alpha(u)}{u} [p_{n+1,i}(u) - p_{ni}(u)]du$$

(where we put $p_{n,n+1}(u) \equiv 0$ and the sum is zero for $n+1 < v'$)

$$\sum_{i=v'}^{n+1}\left[\frac{\lambda_i}{\lambda_{n+1}}\int_0^1 \frac{\alpha(u)}{u}\, p_{n+1,i}(u)\,du - \frac{\lambda_{i+1}}{\lambda_{n+1}}\int_0^1 \frac{\alpha(u)}{u}\, p_{n+1,i+1}(u)\,du\right]$$

$$= \frac{\lambda_{v'}}{\lambda_{n+1}}\int_0^1 \frac{\alpha(u)}{u}\, p_{n+1,v'}(u)\,du.$$

Now by (1.7) and [6], p. 46 (10)

$$\sum_{n=0}^{\infty}\left|\beta_v(n)-\beta_v(n+1)\right| \leqq \int_0^1 \frac{|\alpha(u)|}{u}\left[\sum_{n=v'-1}^{\infty}\frac{\lambda_{v'}}{\lambda_{n+1}}\, p_{n+1,v'}(u)\right]du$$

$$= \int_0^1 \frac{|\alpha(u)|}{u}\,du < \infty,$$

since by [3] Theorem 4.1, for $m \geqq 0$

$$\sum_{n=m}^{\infty}\frac{\lambda_m}{\lambda_n}\, p_{nm}(u) = \begin{cases}1 & 0 < u \leqq 1\\ 0 & u = 0.\end{cases} \tag{2.9}$$

Our theorem now follows by Lorentz's Theorem.

PROOF OF THEOREM 3. Again by Knopp and Lorentz [4] Satz 1, it follows by (1.8) that necessary and sufficient conditions in order that $[QH;\mu_n]$ should be absolutely regular are

$$\sup_{i\geqq 0}\sum_{k=0}^{i}\left|\lambda_{ik}\right| = M < \infty \tag{2.10}$$

and

$$\sum_{k=0}^{i}\lambda_{ik} = 1, \qquad i = 0,1,2,\cdots. \tag{2.11}$$

Now by [1] Sätze 5 and 6, (2.10) is equivalent to the sequence $\{\mu_n\}$ $(n \geqq 0)$ being represented by (1.5) where $\alpha(t)$ is of bounded variation in $[0,1]$. Then by [1] (7), (2.11) is equivalent to $\alpha(1)-\alpha(0)=1$ and this completes our proof.

PROOF OF THEOREM 4. First we have to show that the transformation is well defined for series $\sum_{n=0}^{\infty} a_n$ such that $\lambda_n a_n = \Omega(1)$. Now if

$$\sum_{n=0}^{\infty} \left| \lambda_n a_n - \lambda_{n+1} a_{n+1} \right| < \infty,$$

then $\lambda_n a_n = 0(1)$ and so the existence of the quasi-Hausdorff transform $QH(\alpha)$ of $\sum_{n=0}^{\infty} a_n$ when $\alpha(t)$ satisfies (1.6) and (1.7) was proved in [3] (see the proof of theorem 4.2). By (1.8), (2.4) and [5] p. 46 (11) we have

$$\sigma_n = \sum_{i=0}^{\infty} a_i \sum_{k=0}^{n} \int_0^1 p_{ik}(t) d\alpha(t) \tag{2.12}$$

(where $p_{ik}(t) \equiv 0$ for $i < k$)

$$= \sum_{i=0}^{n} a_i + \sum_{i=n+1}^{\infty} a_i \sum_{k=0}^{n} \int_0^1 p_{ik}(t) d\alpha(t).$$

Therefore if $v' = \max\{n + 1, v\}$ and if $c_n = (1/\lambda_n)$, $n \geqq 1$, we have

$$\beta_v(n) = \sum_{i=v'}^{\infty} \frac{1}{\lambda_i} \sum_{k=0}^{n} \int_0^1 p_{ik}(t) d\alpha(t).$$

Now by (2.7) for every $i \geqq n + 1$,

$$\int_0^1 \sum_{k=0}^{n} p_{ik}(t) d\alpha(t) = \lambda_{n+1} \int_0^1 \frac{\alpha(u)}{u} p_{i,n+1}(t) du. \tag{2.13}$$

Hence for $n \geqq v - 1$ it follows by (2.9) that

$$\beta_v(n) = \int_0^1 \frac{\alpha(u)}{u} \left[\sum_{i=n+1}^{\infty} \frac{\lambda_{n+1}}{\lambda_i} p_{i,n+1}(u) \right] du$$

$$= \int_0^1 \frac{\alpha(u)}{u} du,$$

and consequently

$$\beta_v(n) - \beta_v(n + 1) = 0, \qquad n \geqq v - 1. \tag{2.14}$$

For $n < v - 1$ it follows by (2.13) that

$$\beta_v(n) - \beta_v(n + 1) = \int_0^1 \frac{\alpha(u)}{u} \left[\sum_{i=v}^{\infty} \frac{\lambda_{n+1}}{\lambda_i} p_{i,n+1}(u) - \frac{\lambda_{n+2}}{\lambda_i} p_{i,n+2}(u) \right] du$$

By (2.3)

$$\sum_{=v}^{\infty} \left[\frac{\lambda_{n+1}}{\lambda_i} p_{i,n+1}(u) - \frac{\lambda_{n+2}}{\lambda_i} p_{i,n+2}(u) \right]$$

$$= \sum_{i=v}^{\infty} [p_{i,n+1}(u) - p_{i-1,n+1}(u)]$$

and since by [1] Satz 1

$$\lim_{i\to\infty} p_{i,n+1}(u) = 0, \qquad 0 \leqq u \leqq 1,$$

The second sum is equal to

$$p_{\nu-1,n+1}(u).$$

Hence for $n < \nu - 1$

$$\beta_\nu(n) - \beta_\nu(n+1) = \int_0^1 \frac{\alpha(u)}{u} p_{\nu-1,n+1}(u)du. \tag{2.15}$$

Combining (2.14) and (2.15) we get by (1.7) and [6] p. 46 (10)–(11)

$$\begin{aligned}\sum_{n=0}^{\infty} \left|\beta_\nu(n) - \beta_\nu(n+1)\right| &= \sum_{n=0}^{\nu-2} \left|\beta_\nu(n) - \beta_\nu(n+1)\right| \\ &\leqq \int_0^1 \frac{|\alpha(u)|}{u} \sum_{n=0}^{\nu-2} p_{\nu-1,n+1}(u)du \\ &\leqq \int_0^1 \frac{|\alpha(u)|}{u} du < \infty.\end{aligned}$$

Our theorem now follows by Lorentz's Theorem.

3. A weaker condition. Let $b_0 = 0$ and for $n \geqq 1$

$$b_n = \sum_{k=1}^{n} d_{nk}a_k \text{ where } d_{nk} = \prod_{r=k}^{n} \left(1 + \frac{1}{\lambda_n}\right)^{-1}$$

Then $\{b_n\}$ is the Hausdorff transform by means of the moments $\mu_n = \int_0^1 t^{\lambda_n} dt$, $n \geqq 0$ of the sequence $\lambda_n a_n$. (In the special case $\lambda_n = n$ we have

$$b_n = (a_1 + 2a_2 + \cdots + na_n)/(n+1)).$$

Since $H(\alpha(t) = t)$ is absolutely regular it follows that $\lambda_n a_n = \Omega(1)$ implies $b_n = \Omega(1)$.

It was proved by Tietz [10] §3 that if $\lambda_n a_n = \Omega(1)$ is a Tauberian condition for a method V, then so is $b_n = \Omega(1)$. Thus it follows by our Theorems 1, 2 that

THEOREM 5. *Let the function* $\alpha(t)$ *satisfy* (1.6) *and* 1.7). *Then* $b_n = \Omega(1)$ *is an absolute Tauberian condition for both* $H(\alpha)$ *and* $QH(\alpha)$.

For the Cesàro method (C,α) $\alpha > 0$, Theorem 5 was proved by Hyslop [2]. (See also Maddox [8].).

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UNIVERSITY OF ILLINOIS
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